INVESTIGATION OF COUPLED RADIATION-CONDUCTION HEAT TRANSFER IN CYLINDRICAL SYSTEMS BY DISCONTINUOUS SPECTRAL ELEMENT METHOD

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ABSTRACT

Nowadays, in order to obtain a higher efficiency in aero-engines, the increase in the turbine inlet temperature of gas turbine engine is an urgent need. At present, the turbine inlet temperature is close to 2000K, which means the radiation and coupled radiation-conduction heat transfer play more and more important roles in hot section of aero-engines. As we all know, considering the cylindrical symmetry of the aero-engine. It is convenient to adopt the cylindrical coordinate to simplify the systems in aero-engines with cylindrical geometry, such as annular combustor and exhaust nozzle. In this paper, the Discontinuous Spectral Element Method (DSEM) is extended to solve the radiation and coupled radiation-conduction heat transfer in cylindrical coordinate systems. Both the spatial and angular computational domains of radiative transfer equation (RTE) are discretized and solved by DSEM. For coupled radiation-conduction heat transfer problem, Discontinuous Spectral Element Method- Spectral Element Method (DSEM-SEM) scheme is used to avoid using two sets of computational grids which would cause the increase of computational cost and the decrease of accuracy. Then, the effects of various geometric and thermal physical parameters are comprehensively investigated. Finally, these methods are further extended to 2D cylindrical system.

INTRODUCTION

The higher turbine inlet temperature means higher thermal efficiency and power output of turbine. Therefore, the ability to operate at higher temperature has been an important factor in improving the performance of aero-engines. Over the past few decades, the turbine inlet temperature has increased from 1000K in the first generation to more than 2000K (Yin and Rao 2020) in the latest generation aero-engines. The increase of temperature and the fact that total amount of radiation rises as the fourth power of the absolute temperature (Howell 2015) determine that the radiation and coupled radiation-conduction heat transfer play a more and more important role in hot section of aero-engines.

The radiation and coupled radiation-conduction problems in aero-engines involve two equations: radiative transfer equation (RTE) and energy equation. Considering the cylindrical symmetry of the aero-engine, these two governing equations can be expressed in more simple way using cylindrical coordinates which could bring a lot convenience like the implement of boundary conduction, etc. Different from conduction and convection which only spatial dimension need to be concerned, because of the directional nature of radiation, the angular distribution of radiation also need to be calculated for radiation heat transfer problem. This angular dependence adds to more dimensions i.e. polar angle and azimuthal angle. This high dimensionality of radiation heat transfer would lead to great challenge to the numerical computation. Besides, RTE, the governing equation of radiation heat transfer, can be considered as a kind of convection-dominated equation from the point of numerical computation. The characteristic of convection-dominated means that RTE suffer from difficult problems to gain a stable and accurate numerical solution. The characteristics of radiation introduced above determine that
solving RTE and obtaining the radiation distribution are the most important and difficult part in both radaiton and coupled radiation and conduction heat transfer problems.

In recent years, the radiation and coupled radiation-conduction heat transfer in cylindrical coordinate has evoked wide interests of many researchers. As early as 1982, Fernandes and Francis (1982) gave the rigorous formulations of combined conduction and radiation in concentric cylinders and solved it by Galerkin finite element method. Pandey (1989) employed undetermined parameters method to solve this coupled problem for gray and nongray gases contained between infinitely long concentric cylinders with black surfaces. Aouled-Dlala et al. (2007) investigated coupled radiation-conduction heat transfer in gray hollow spheres and cylinders. They used finite Chebyshev transform (FCT) to improve the performance of discrete ordinates method, and adopted Chebyshev polynomials to approximate the angular derivative term instead of finite difference scheme. The FCT is more accurate than traditional discrete ordinate method. Mishra and Krishna (2011) developed the modified discrete ordinate method (MDOM) and lattice Boltzmann method to analyze coupled radiative-conductive heat transfer in infinite and finite concentric cylinders with absorbing, emitting, and scattering medium.

In this paper, the discontinuous spectral element method (DSEM) is adopted to solve the radiation and coupled radiation-conduction heat transfer in cylindrical coordinate systems. For the numerical solution of RTE, the space-angle scheme is used to deal with the lack of angular resolution problem which exists in the results of RTE. Space-angle scheme means both the spatial and angular computational domains of RTE are discretized and solved by DSEM. The results demonstrate that the space-angle DSEM performs much better than the traditional hybrid methods and exhibit exponential convergence in both spatial and angular domain. For coupled radiation-conduction heat transfer problem, spectral element method (SEM) is adopted to solve the energy equation after the radiation distribution is obtained by DSEM and then substituted into the energy equation as radiative source term. This kind of DSEM-SEM scheme could avoid using two sets of computational grids which would cause the increase of computational cost and the decrease of accuracy. And the results show that this DSEM-SEM scheme is feasible to solve the coupled radiation and conduction heat transfer problem. Then, the effects of various geometric and thermal physical parameters are comprehensively investigated. Finally, these methods are further extended to 2D cylindrical system.

The paper is organized as follow. In section 2 methodology, RTE in concentric cylindrical medium and numerical method are introduced. In section 3, the numerical results of radiation and coupled radiation-conduction heat transfer in both 1D and 2D cylindrical system are presented and analyzed. Finally, conclusions are summarized in section 4.

METHODOLOGY

RTE in Concentric Cylindrical Medium

As shown in Fig. 1, the present paper studies the radiation and coupled radiation and conduction heat transfer in cylindrical coordinate system. The $\theta$ is the polar angle and $\phi$ is azimuthal angle. The absorbing, emitting and anisotropic scattering medium is filled in concentric cylinders where the subscripts $in$ and $out$ refer to the inner and outer wall respectively.

The RTE in cylindrical gray medium can be written as Modest (2013),

$$
\frac{\mu}{r} \frac{\partial [rI(r, \Omega)]}{\partial r} - \frac{1}{r} \frac{\partial [\eta I(r, \Omega)]}{\partial \phi} + \beta I(r, \Omega) = S(r, \Omega) \tag{1}
$$

with the boundary conditions:

$$
I(R_{in}, \Omega) = \varepsilon_{in}I_{p}(R_{in}) + \frac{(1 - \varepsilon_{in})}{\pi} \int_{\mu' < 0} I(R_{in}, \Omega') |\mu'| d\mu' \quad \mu > 0 \tag{2a}
$$

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$$
\frac{\mu}{r} \frac{\partial [rI(r, \Omega)]}{\partial r} - \frac{1}{r} \frac{\partial [\eta I(r, \Omega)]}{\partial \phi} + \beta I(r, \Omega) = S(r, \Omega) \tag{1}
$$

with the boundary conditions:

$$
I(R_{in}, \Omega) = \varepsilon_{in}I_{p}(R_{in}) + \frac{(1 - \varepsilon_{in})}{\pi} \int_{\mu' < 0} I(R_{in}, \Omega') |\mu'| d\mu' \quad \mu > 0 \tag{2a}
$$

Figure 1 Physical model concentric cylinders.
\[ I(R_{out}, \Omega) = \varepsilon_{out} I_b(R_{out}) + \frac{(1 - \varepsilon_{in})}{\pi} \int_{\mu > 0} I(R_{out}, \mu') |\mu'\rangle\langle \mu'| \mu < 0 \]  

(2b)

where \( I(r, \Omega) \) is the radiative intensity at spatial position \( r \) along angular direction \( \Omega \); \( \Omega(\theta, \phi) = e_r \mu + e_\theta \eta + e_\phi \xi = e_r \sin \theta \sin \phi + e_\theta \sin \theta \cos \phi + e_\phi \cos \theta \) is the angular direction described by polar angle \( \theta \) and azimuthal angle \( \phi \); \( \beta = \kappa_a + \kappa_s \) is the extinction coefficient in which \( \kappa_a \) and \( \kappa_s \) are absorption coefficient and scattering coefficient, respectively. \( S(r, \Omega) \) is the source term which is defined as,

\[ S(r, \Omega) = \kappa_a I_b + \frac{\kappa_s}{4\pi} \int_{4\pi} I(r, \Omega') \Phi(\Omega, \Omega') d\Omega' \]  

(3)

where \( I_b \) is the black body radiative intensity. The scattering phase function \( \Phi(\Omega, \Omega') \) describes the probability of photon scattering from incident direction \( \Omega \) to the direction \( \Omega' \).

In this paper, the symmetry of one-dimensional cylinder is used. Therefore, the domain considered here is,

\[ r \in [R_{in}, R_{out}], \quad \phi \in [0, \pi], \quad \theta \in [0, \frac{\pi}{2}] \]  

(4)

**Discontinuous Galerkin Method**

The discontinuous Galerkin method (DG method) Cockburn (2003) combines features of the finite element and the finite volume framework. Unlike traditional continuous Galerkin methods, the DG method works over a trial space of functions that are only piecewise continuous. This means the solution can possibly have a different values on the shared boundary of different elements. An alternative formulation, the so-called weak formulation, is used to get a representation as a piecewise polynomial, in which the polynomials are discontinuous at the element boundaries. The transport equation is taken as an example to introduce the implementation of DG.

\[ \nabla \cdot (au) = 0 \quad x \in \mathbb{R}^n \]  

(5)

First, the solution domain \( \mathbb{R}^n \) is decomposed into \( N_{el} \) non-overlapping elements \( E_{el}^n \):

\[ E_i^n \cap E_j^n = \emptyset \quad i \neq j \quad \mathbb{R}^n = \bigcup_{el=1}^{N_{el}} E_{el}^n \]  

(6)

Then, we need to derive the weak form of the PDE Eq.5. Multiply both sides of Eq.5 by a differentiable test function \( v \), such as polynomials for example, and integrate the resulting equation over the space \( E \) to get,

\[ \int_E (\nabla \cdot (au)) v = 0 \]  

(7)

using integration by parts,

\[ \int_E u v - \int_E (au) \cdot \nabla v + \int_{\partial E} (au) \cdot n v ds = 0 \]  

(8)

where \( \partial E \) means the boundary of element \( E \), \( n \) denotes the outward normal vector of the boundary. And Eq.8 is referred to as a weak formulation of Eq.5.

The calculation of DG is carried out on each element, and DG establishes a link between the values of solution in different elements only through flux on the boundary. Therefore, the numerical flux \( \overline{au} \) instead of \( au \) need to be introduced to exchange the information between different elements. In this paper, the classical up-winding scheme is considered, in which this numerical flux is modeled as,

\[ \overline{au} = a \{ u \} + |a \cdot n| [u] \]  

(9)

where the operator \( \{ \cdot \} \) and \( [\cdot] \) denote the mean value and the jump value of arguments across element boundary respectively.

\[ \{ u \} = \frac{1}{2} (u^+ + u^-) \quad [u] = \frac{1}{2} (u^+ - u^-) \]  

(10)

Here the superscript operator \(^+\) and \(^-\) denote the values at the boundary inside and outside element, respectively.
Spectral Element Discretization

In spectral element discretization (Patera 1984), the nodal basis functions on each element are constructed by Chebyshev polynomial expansion. For one-dimensional case in standard computing domain \([-1,1]\), the nodal basis functions are Lagrange interpolation polynomials through the Chebyshev–Gauss–Lobatto points \(\alpha^{CGL}\),

\[
\alpha^{CGL}_j = \cos\left(\frac{j-1}{N-1}\pi\right), \quad j = 1, 2, \ldots N
\]  

(11)

After the construction of basis functions, quantity \(u\) to be solved in any position on element can be approximated as,

\[
u = \sum_{i=1}^{N_e} u\Gamma_i
\]

(12)

where \(N_e\) is the number of solution nodes in element \(E\). The \(\Gamma_i\) is the the basis function of node \(i\).

Discontinuous Spectral Element Method Discretization of RTE

In order to use DSEM to discretize RTE in cylindrical coordinate system, it is necessary to use the mathematically equivalent form instead of the usual form of the RTE given by Eq.1. However, as shown in the Eq.1, there exist a minus sign ‘\(-\)’ before the angular differential term \(\frac{\partial}{\partial \varphi}\) in the RTE. This means there are two different mathematical equivalent forms for the Eq.1. It is worth noting that it is critical to choose the proper mathematical equivalent form for the successful implement of DSEM. The two different kind mathematical equivalent form of Eq.1 are:

- **Forma**
  \[
  \frac{\partial}{\partial r}(r\mu I) + \frac{\partial}{\partial \varphi}(-\eta I) + \beta rI = rS
  \]  

(13a)

- **Formb**
  \[
  \frac{\partial}{\partial r}(r\mu I) - \frac{\partial}{\partial \varphi}(\eta I) + \beta rI = rS
  \]

(13b)

where the dependences of \(I\) on \(r\) and \(\mu\) have been omitted to simplify the expression.

**Physical meaning of the two mathematical equivalent**

As mentioned before, the computational domain in this paper is: \(r \in [R_{in}, R_{out}]\), \(\varphi \in [0, \pi]\), \(\theta \in [0, \frac{\pi}{2}]\), which means \(\eta = \sin \theta \cos \varphi \geq 0\) in the whole computational domain. After applying the DSEM to **Form a** (Eq.13a) and **Form b** (Eq.13b) respectively, we can obtain:

- **Forma**
  \[
  \int_E I(-\mu r \frac{\partial I}{\partial r} - (-\eta) \frac{\partial I}{\partial \varphi} + \beta rI) drd\varphi d\theta + \int_{\partial E} \mathbf{n}_\varphi \cdot (\mu r I) \Gamma d\varphi + \int_{\partial E} \mathbf{n}_\varphi \cdot (-\eta I) \Gamma d\varphi = \int_E rS \Gamma d\varphi d\theta
  \]  

(14a)

- **Formb**
  \[
  \int_E I(-\mu r \frac{\partial I}{\partial r} + (\eta) \frac{\partial I}{\partial \varphi} + \beta rI) drd\varphi d\theta + \int_{\partial E} \mathbf{n}_\varphi \cdot (\mu r I) \Gamma d\varphi - \int_{\partial E} \mathbf{n}_\varphi \cdot (\eta I) \Gamma d\varphi = \int_E rS \Gamma d\varphi d\theta
  \]

(14b)

It is obvious that the only difference between the two mathematical equivalent forms lies in the boundary integral \(\pm \int_{\partial E} \mathbf{n}_\varphi \cdot (\mp \eta I) \Gamma d\varphi\). Therefore, the influence of different equivalent forms on boundary integral \(\pm \int_{\partial E} \mathbf{n}_\varphi \cdot (\mp \eta I) \Gamma d\varphi\) need to be studied.

First, we consider the influence of **Form a**. For the case when \(\mathbf{n}_\varphi = -1\) (Fig.2(a)), using up-winding scheme and substituting Eq.9, Eq.10 and \(\mathbf{n}_\varphi = -1\) into boundary integral, we can obtain,

\[
\int_{\partial E} \mathbf{n}_\varphi \cdot (-\eta) \Gamma d\varphi = \int_{\partial E} (1) \cdot (-\eta) \Gamma d\varphi + \int_{\partial E} (1) \cdot (-\eta) \Gamma d\varphi + \int_{\partial E} (1) \cdot (-\eta) \Gamma d\varphi = \int_{\partial E} \mathbf{n}_\varphi \cdot (-\eta) \Gamma d\varphi
\]  

(15)
The equation above means that the $I^+$ inside the element $E$ as shown in Fig.2(a) is the upstream physical quantity. That is to say that the radiative intensity with larger $\phi$ is the upstream and the radiative intensity with smaller $\phi$ is the downstream. The radiation propagate from the larger $\phi$ to the smaller $\phi$.

For the case when $n_\phi = 1$ (Fig.2(b)), Substituting Eq.9, Eq.10 and $n_\phi = 1$ into boundary integral, we can obtain,

$$\int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma = \int_{\partial E} (1) \cdot (\eta) I \, d\Gamma + \int_{\partial E} (1) \cdot (I I) \, d\Gamma = \int_{\partial E} (1) \cdot (\eta) I \, d\Gamma = \int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma \tag{16}$$

The equation above means that the $I^−$ outside the element $E$ as shown in Fig.2(b) is the upstream physical quantity. In other words, the radiation transfer from the larger $\phi$ to the smaller $\phi$. In conclusion, the Physical meaning of Form $a$ is that the radiation propagate from the larger $\phi$ to the smaller $\phi$ in $\phi$ direction.

Similarly, the physical meaning of Form $b$ can be analyzed in this way. For the case when $n_\phi = -1$, we can get,

$$-\int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma = -\int_{\partial E} (1) \cdot (\eta) I \, d\Gamma + \int_{\partial E} (1) \cdot (\eta) I \, d\Gamma = -\int_{\partial E} (1) \cdot (\eta) I \, d\Gamma = -\int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma \tag{17}$$

The equation above means that the $I^−$ outside the element is the upstream physical quantity. That is to say that the radiative intensity with smaller $\phi$ is the upstream while larger $\phi$ is the downstream. The radiation propagate from the smaller $\phi$ to the larger $\phi$.

When $n_\phi = 1$, we will get,

$$-\int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma = -\int_{\partial E} (1) \cdot (\eta) I \, d\Gamma + \int_{\partial E} (1) \cdot (\eta) I \, d\Gamma = -\int_{\partial E} (1) \cdot (\eta) I \, d\Gamma = -\int_{\partial E} \mathbf{n}_\phi \cdot (\eta) I \, d\Gamma \tag{18}$$

The equation above means that the $I^+$ inside the element is the upstream physical quantity in $\phi$ direction. That is to say that the radiation propagate from the smaller $\phi$ to the larger $\phi$.

In conclusion, the physical meaning of Form $b$ is that the radiation propagate from the smaller $\phi$ to the larger $\phi$ in $\phi$ direction. This conclusion means that the physical meanings of Form $a$ and Form $b$ are just the opposite. Therefore, we need to reveal the real physical scene of radiation propagation in the cylindrical coordinate system, and then to determine which mathematical equivalent form should be chosen in this paper.

**Radiation propagation in the cylindrical coordinate system**

In the angular direction $\phi$, there exists an implicit relationship between the “upstream” and “downstream”. The angular computational domain in the cylindrical symmetric RTE is $\phi \in [0, \pi]$. Fig.3(a) and 3(b) present two typical propagation of the radiation to clarify the implicit relationship of the “upstream” and “downstream” in angular $\phi$ direction. It is obvious that the position with larger $\phi$ is the “upstream” while the smaller $\phi$ is the “downstream”, the radiation propagates from the larger $\phi$ to smaller $\phi$.

It can be seen that the physical meaning of Form $a$ is consistent with the physical reality of radiation propagation in the cylindrical coordinate system while the physical meaning of Form $b$ is just the opposite. Therefore, the Form $a$ (Eq.14a) is the mathematical equivalent which we should choose in this paper while the Form $b$ is not allowed to be used.

**Spectral Element Method Discretization of Energy Equation**

For coupled radiation and conduction heat transfer problem in cylindrical coordinate system, different from the RTE which involving three-dimensional $r - \phi - \theta$ domain, the one-dimension energy equation with radiative source term is,

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} = \frac{4\pi \kappa_a}{k} \left( I_b (T^* (r)) - \frac{1}{4\pi} G(r) \right) \tag{19}$$

where $k$ is thermal conductivity, $T^*$ is the temperature value of the last iteration step.

Since the coupling between the energy equation and the equation of radiation transfer is highly nonlinear due to the relationship between $I_b$ and the fourth power of temperature $T$, it is necessary to employ iterative methods to solve the coupled radiation and conduction problem.

The energy equation Eq.20 is a classical elliptic equation. Therefore, a good result can be obtained by directly using the spectral element method (SEM) without introducing discontinuous Galerkin scheme. The mesh of SEM adopts the same number of elements and nodes as DSEM to avoid using two sets of computational grids which would cause the increase of computational cost and the decrease of accuracy. Applying the SEM to Eq.20, we can obtain:

$$-\int_E \frac{dT}{dr} \frac{dT}{dr} \, dr + \int_{\partial E} \frac{dT}{dr} \left( \mathbf{n}_r \cdot 1 \right) \, d\Gamma + \int_{\partial E} \frac{1}{r} \frac{dT}{dr} \, d\Gamma = \int_E \frac{4\pi \kappa_a}{k} \left( I_b (T^* (r)) - \frac{1}{4\pi} G(r) \right) \, d\Gamma \tag{20}$$
The Extension to 2D Cylindrical System

Fig.4 depicts the 2D cylindrical system, the $z$ direction need to be considered compared with 1D cylinder. Therefore, from the mathematical point of view, the radiative heat transfer problem in 2D cylindrical system considered here is actually 4-D case, the polar angle $\theta$, azimuthal angle $\varphi$, spatial $r$ and $z$.

![Figure 4 Physical model of 2D concentric cylinders.](image)

Radiation propagation in the 2D cylindrical system

The RTE in 2D cylindrical gray medium is,

$$\mu \frac{\partial [rI(r,z,\Omega)]}{\partial r} - \frac{1}{r} \frac{\partial [\eta I(r,z,\Omega)]}{\partial \varphi} + \frac{\xi I(r,z,\Omega)}{z} + \beta I(r,z,\Omega) = S(r,z,\Omega)$$  \hspace{1cm} (21)

The boundary conditions of gray walls we considered in this paper are:

$$I(R_{in},z,\Omega) = \varepsilon_{in} I_{b}(R_{in}) + \left(1 - \varepsilon_{in}\right) \frac{\pi}{\mu} \int_{\mu' < 0} I(R_{in},z,\Omega') |\mu'| d\mu' \quad \mu > 0$$  \hspace{1cm} (22a)

$$I(R_{out},z,\Omega) = \varepsilon_{out} I_{b}(R_{out}) + \left(1 - \varepsilon_{in}\right) \frac{\pi}{\mu} \int_{\mu' > 0} I(R_{out},z,\mu') |\mu'| d\mu' \quad \mu < 0$$  \hspace{1cm} (22b)

$$I(r,Z_{down},\Omega) = \varepsilon_{down} I_{b}(Z_{down}) + \left(1 - \varepsilon_{down}\right) \frac{\pi}{\xi} \int_{\xi' < 0} I(r,Z_{down},\Omega') |\xi'| d\xi' \quad \xi > 0$$  \hspace{1cm} (22c)
\[ I(r, Z_{up}, \Omega) = \varepsilon_{up} I_{b}(Z_{up}) + \frac{(1 - \varepsilon_{up})}{\pi} \int_{\xi' > 0} I(r, Z_{up}, \Omega') |\xi'| d\xi' \quad \xi' < 0 \] (22d)

There also exists the symmetry for 2D cylinder and the domains we need to considered are,
\[ r \in [R_{in}, R_{out}], \quad z \in [Z_{down}, Z_{up}], \quad \varphi \in [0, \pi], \quad \theta \in [0, \pi] \] (23)

As the analysis in 1D cylinder case, for the successful implementation of DSEM, the mathematical equivalent form of Eq. 21 we need to choose is,
\[ \frac{\partial}{\partial r} (r \mu I) + \frac{\partial}{\partial \varphi} (-\eta I) + \frac{\partial}{\partial z} (\xi r I) + \beta r I = r S \] (24)

And the corresponding formula after applying DSEM to Eq. 24 is,
\[ \int_{E} I[ -\mu r \frac{\partial \Gamma}{\partial r} - ( - \eta) \frac{\partial \Gamma}{\partial \varphi} - \xi r \frac{\partial \Gamma}{\partial z} + \beta r \Gamma] d\varphi d\theta dz + \int_{\partial E} n_{r} (r \mu) \Gamma d\varphi + \int_{\partial E} n_{\varphi} ( -\eta) \Gamma d\theta + \int_{\partial E} n_{z} (\xi r) \Gamma d\theta \]
\[ = \int_{E} r S d\varphi d\theta dz \] (25)

**Coupled Radiation and Conduction Problem in the 2D cylindrical system**

The coupled radiation and conduction heat transfer problem in 2D cylindrical system involving two-dimensional \( r - z \) domain, the 2D energy equation with radiative source term is,
\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{4\pi K_{0}}{k} \left( I_{b}(T^{*}(r, z)) - \frac{1}{4\pi} G(r, z) \right) \] (26)

After applying the SEM to Eq. 26, we would obtain:
\[ - \int_{E} \frac{dT}{dr} \frac{d\Gamma}{dr} r dr dz + \int_{\partial E} \frac{dT}{dr} \Gamma (n_{r}, [1, 0]) dl + \int_{E} \frac{1}{r} \frac{dT}{dr} \Gamma dz dr + \int_{\partial E} \frac{dT}{dz} \Gamma (n_{z}, [0, 1]) dl = \int_{E} \frac{4\pi K_{0}}{k} \left( I_{b}(T^{*}(r, z)) - \frac{1}{4\pi} G(r, z) \right) \Gamma dz dr \] (27)

**RESULTS AND DISCUSSION**

**Grid Independence Test**

The grid independence test are conducted in this section and the radiative equilibrium is considered firstly. The radius ratio of concentric cylinder is \( R_{in}/R_{out} = 0.1 \), the temperature ratio \( T_{in}/T_{out} = 0 \), which means both the wall are cold. The cylinder is filled with isotropically scattering medium, because of the isotropic scattering, the scattering albedo does not change the distribution of radiation heat flux. Fig. 5(a) gives the change of the dimensionless radiative heat flux \( q_{r}^{*} = q_{r}/\sigma T_{m}^{4} \) at the inner wall with the extinction coefficient \( \beta \) when the number of nodes in the element \( M_{r} \times M_{\varphi} \times M_{\theta} \) is different and the number of elements is fixed as \( N_{r} \times N_{\varphi} \times N_{\theta} = 8 \times 8 \times 8 \). The results in Fig. 5(a) shows that when \( M_{r} \times M_{\varphi} \times M_{\theta} \) is larger than \( 8 \times 8 \times 8 \), the dimensionless radiative heat flux \( q_{r}^{*} \) at the inner wall remains almost unchanged and in consistent with the result of the modified discrete coordinate method (MDOM) Mishra and Krishna (2011). Fig. 5(b) presents the changing trend of the \( q_{r}^{*} \) at the inner wall with the extinction coefficient under the condition that the number of nodes in the element \( M_{r} \times M_{\varphi} \times M_{\theta} = 8 \times 8 \times 8 \) remains constants and the number of elements \( N_{r} \times N_{\varphi} \times N_{\theta} \) changes. It can be seen that, the results are not in good agreement with the results in reference when \( N_{r} \times N_{\varphi} \times N_{\theta} = 2 \times 2 \times 2 \). With the number of elements increase from \( 3 \times 3 \times 3 \) to \( 6 \times 6 \times 6 \), the results agree well with that in reference and remain basically unchanged.

**Radiative Heat Transfer in 1D cylindrical system**

Note that the radiative equilibrium problem is already considered in the grid independence above, then another typical radiative transfer problem, non-radiative equilibrium problem would be numerically analyzed in the following section.

For non-radiative equilibrium problem, the radiative participating medium is the radiation sources instead of the cold concentric cylinder boundaries \( T_{in} = T_{out} = 0 \). The change of the dimensionless radiative heat flux \( q_{r}^{*} \) at both inner and outer wall with the extinction coefficient \( \beta \) are depicted in Fig. 6. In Fig. 6(a), the scattering albedo \( \omega = 0 \). And it can be seen that for a given value of \( \beta \), the the dimensionless radiative heat flux \( q_{r}^{*} \) increases with the increase of the radius ratio \( R_{in}/R_{out} \) in both inner and outer wall. In Fig. 6(b) \( q_{r}^{*} \) decrease all the time when the scattering albedo \( \omega \) becomes larger. This is because the larger \( \omega \) means more energy the medium would be scattered, and the radiative heat flux would decrease consequently. It is obvious that the numerical results of DSEM are in good agreement with the results in reference Mishra and Krishna (2011) for the non-radiative equilibrium problem.
(a) The number of nodes in the element $M_r \times M_\phi \times M_\theta$

(b) The number of elements $N_r \times N_\phi \times N_\theta$

**Figure 5** The grid independence test.

(a) Different radius ratio $R_i/R_{\text{out}}$

(b) Different scattering albedo $\omega$

**Figure 6** The influence of extinction coefficient $\beta$ on dimensionless radiative heat flux $q_\ast^\prime$.  

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Table 1 Values of the dimensionless total heat flux in inner wall $q'_t(R_{in})$ and outer wall $q'_t(R_{out})$ at the boundaries obtained by DOM, FCT Aouled-Dlala et al. (2007) and SCM

<table>
<thead>
<tr>
<th>$N_{cr}$</th>
<th>$\omega$</th>
<th>$q'<em>t(R</em>{in})$</th>
<th>$q'<em>t(R</em>{out})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>FCT</td>
<td>CSM</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5</td>
<td>1.6436</td>
<td>1.6421</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
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<td>1.6468</td>
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<td>0.5</td>
<td>3.5045</td>
<td>3.4840</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>3.5529</td>
<td>3.5271</td>
</tr>
<tr>
<td>0.01</td>
<td>0.5</td>
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<tr>
<td>0.1</td>
<td>0.223172</td>
<td>22.0889</td>
<td>21.8675</td>
</tr>
</tbody>
</table>

Coupled Radiation-Conduction Heat Transfer in 1D cylindrical system

Table 1 shows the dimensionless total heat flux in inner wall $q'_t(R_{in})$ and outer wall $q'_t(R_{out})$ under the conditions of different thermal conductivity-emissivity $N_{cr}$ and different scattering albedo $\omega$ for the case $R_{in}/R_{out} = 0.5$, $\varepsilon_{in} = \varepsilon_{out} = 1$, $T_{out}/T_{in} = 0.5$ and $\beta = 1$. The mesh used in this case is $N_r \times N_\phi \times N_\theta = 16 \times 8 \times 8$ and $M_r \times M_\phi \times M_\theta = 5 \times 5 \times 5$. It can be seen that the numerical results of the DSEM-SEM used in this paper are basically consistent with the results of CSM and FCT. Thus, it is verified that the DSEM-SEM can correctly solve the coupled radiation-conduction problem in one-dimensional cylindrical system.

After verifying the feasibility of the DSEM-SEM scheme, we study the effects of conduction-radiation parameter $N_{cr}$ on dimensionless temperature for coupled radiation-conduction problem. Fig. 7 gives the dimensionless temperature distribution for different $N_{cr}$ and the case of pure heat conduction. It can be seen that, when $N_{cr}$ is relatively small, the dimensionless temperature has an obvious nonlinear distribution. However, with the further increase of $N_{cr}$, the dimensionless temperature distribution tends to be linear and gets closer to the result of pure heat conduction.

The conduction-radiation parameter $N_{cr} = k\beta / 4\sigma T_{ref}^3$ is defined as the ratio of conductive heat transfer and radiation heat transfer. It is obvious that, for the large value of $N_{cr}$, conduction plays a dominant role in this coupled heat transfer problem. On the contrary, the small value of means radiation becomes much more pronounced. For the conduction-dominated problem, the dimensionless temperature tends to linear while the dimensionless temperature tends to nonlinear for the radiation-dominated problem. Thus, with the increasing of conduction-radiation parameter, the distribution of dimensionless temperature changes from nonlinear to linear.

Figure 7 Effects of conduction-radiation parameter $N_{cr}$ on dimensionless temperature distribution.
The Radiation and Coupled Radiation-Conduction Heat Transfer in 2D cylindrical system

The non-radiative equilibrium is considered in the 2D cylindrical system. The inner, outer, up and down walls of the 2D cylinder are cold, \( T_{\text{wall}} \equiv 0 \). And all of the four walls are black diffusive boundary. The height of the cylinder is \( H = Z_{\text{up}} - Z_{\text{down}} = 2 \), \( R_{\text{out}} = 1 \) the radius ratio \( R_{\text{in}}/R_{\text{out}} = 0.5 \). Fig. 8 shows the dimensionless radiative heat flow along the axial \( z \) direction at the outer wall of a two-dimensional cylinder with various radiative heat transfer parameters.

Fig. 8 shows the results of DSEM of the 2D cylinder case, the results are compared with the results of improved discrete coordinate method (MDOM) in ref. Mishra et al. (2011). The comparison shows that in different cases, the results of DSEM are in good agreement with the results in reference, which verifies the feasibility of DSEM in two-dimensional cylindrical system. The effects of different radiative heat transfer parameters on dimensionless radiative heat flux \( q_{r}^* \) will be further analyzed.

Fig. 8(a) shows the influence of extinction coefficient \( \beta \) on \( q_{r}^* \) when the scattering albedo \( \omega = 0 \) and radius ratio is set to \( R_{\text{in}}/R_{\text{out}} = 0.5 \). It can be seen from the figure that with the increase of extinction coefficient \( \beta \), the \( q_{r}^* \) increases significantly. This is because when the scattering albedo \( \omega \) is fixed, the larger the extinction coefficient \( \beta \) means there would be more interaction between photon and medium. Therefore, the radiative heat flux \( q_{r}^* \) would increases with the increase of \( \beta \).

The influence of scattering albedo \( \omega \) on dimensionless radianve heat flux \( q_{r}^* \) is given in Fig. 8(b), in which the extinction coefficient \( \beta \) and radius ratio \( R_{\text{in}}/R_{\text{out}} \) remain unchanged. The scattering albedo \( \omega \) represents the degree of the scattering. As scattering albedo approach 1, the scattering become more stronger, and the medium would absorb less radiative energy. Therefore, when other parameters remain unchanged, the increase of scattering albedo \( \omega \) would lead to the decrease of radiative heat flow. And this trend is reflected in Fig. 8(b) clearly. Compared with the case without scattering \( \omega = 0 \), a significant reduction of dimensionless radianve heat flux \( q_{r}^* \) is observed when \( \omega = 0.9 \).

Coupled Radiation-Conduction Heat Transfer in 2D cylindrical system

In this chapter I would extend the DSEM-SEM scheme from 1D cylinder case to solve the coupled radiation-conduction problem in 2D cylindrical system. The results solved by Finite Difference Method-Finite Volume Method (FDM-FVM) in ref. Mishra et al. (2011) are also presented for comparison, in which the radiative intensity are obtained by FVM while the energy equation is solved by FDM.

In this case, the height of 2D cylinder \( H = Z_{\text{up}} - Z_{\text{down}} = 1 \), \( R_{\text{out}} = 1 \), the medium inside cylinder is isotropic scattering. Radiative heat transfer parameters are: extinction coefficient \( \beta = 1 \), scattering albedo \( \omega = 0 \). The outer and up wall are cold wall \( T_{\text{out}} = T_{\text{up}} = 0 \) while \( T_{\text{down}} = 100 = T_{\text{ref}} \), the inner wall is adiabatic boundary, which means \( \partial T / \partial r = 0 \).

Fig. 9 shows the influence of radius ratios \( R_{\text{in}}/R_{\text{out}} \) on dimensionless temperature \( \Theta \) of 2D concentric cylinder when the conduction-radiation parameter \( N_{\text{cr}} = 0.1 \) remains unchanged. The temperature distribution at the axial central plane \( z = (Z_{\text{down}} + Z_{\text{up}})/2 \) is given. It can be seen that as \( R_{\text{in}}/R_{\text{out}} \) decreases, the dimensionless temperature at the axial central plane gradually increases. This is because the space inside the cylinder becomes larger when \( R_{\text{in}}/R_{\text{out}} \) decreases, and more heat can be transferred from the down wall \( T_{\text{down}} \) with high temperature to the axial central plane, which causes the increase of dimensionless temperature in this plane. And it can be expected that the temperature distribution would reach the highest when the \( R_{\text{in}}/R_{\text{out}} = 0 \) (i.e. a solid 2D cylinder), and the corresponding \( \Theta \) at \( r^* = 0 \) in axial central plane would be 0.5.
CONCLUSIONS

In this paper, two different kinds of mathematical equivalent forms of RTE in cylindrical coordinate systems are studied and compared. Through the analysis from the perspective of physical meaning, the correct form of RTE in cylindrical coordinate systems used for DSEM are given. Then, both the spatial and angular domain are discretized by DSEM to solve the radiative heat transfer problem in cylindrical systems. Besides, considering the numerical characteristics of both RTE and energy equations, DSEM-SEM scheme are proposed to solve the coupled radiation-conduction problem. Then, several cases are given and analyzed to prove the feasibility for solving radiation and coupled radiation-conduction heat transfer in cylindrical systems. Finally, these methods are further extended to 2D cylindrical systems which is closer to the real case for the gas turbine.

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