

On the Three-Dimensional Body Force Model

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ABSTRACT

This paper presents the derivation of a steady-state three-dimensional throughflow model. The unique feature of this throughflow model compared to conventional models is its capability of modelling three-dimensional flows such as those under inlet distortions. With proper averaging operators, three-dimensional throughflow equations can be obtained from the unsteady Reynolds Averaged Navier-Stokes (URANS) equation. This process yields body force terms in equations that account for the existence of blade rows in the flow field. The energy equation is also addressed by an alternative definition of stagnation enthalpy in throughflow equations. Finally, due to the three-dimensional nature of this model, a matching loss model accounting for the viscous effects in flows over blade passages is also developed based on the conventional axisymmetric loss model.

INTRODUCTION

The early-stage throughflow used simplified radial-equilibrium analysis in the 1940s (Smith, 1966, Cumpsty, 1989). Later more sophisticated throughflow models based quasi-3D theory from Wu (1952), such as streamline curvature and body force models, have been widely used in various applications in turbomachinery in the past few decades (Casey and Robinson, 2010; Damle et al., 1997; Denton, 1978; Jennions and Stow, 1985a, 1985b; Novak 1967; Spurr, 1980; Sturmayer and Hirsch, 1999), covering the areas of flow field predictions (Casey and Robinson, 2010; Novak 1967; Spurr, 1980; Sturmayer and Hirsch, 1999), inverse designs (Damle et al., 1997; Jennions and Stow, 1985b), quasi-3D modeling (Denton, 1978; Jennions and Stow, 1985a, 1985b) etc.

Most applicable throughflow models were developed in the relative frame of motion (frame that moves with rotating blade rows). The flow fields within a blade passage can be approximated as steady in the relative frame and it yields axisymmetric and steady form of governing equations, which is a huge bonus to the reduction of computational costs. Such an approximation, however, implies that the inflow is in general clean (uniform) and hence the flow field within one

blade passage can represent that in the full wheel. In other words, these throughflow models are not theoretically adequate in addressing inlet distortion problems, where the flow field varies from one passage to another and it can hardly be approximated as steady in the relative frame.

The goal of this work is to derive a throughflow model that can be readily used for general 3D flow fields. The basic philosophy of this work is inherited from Adamczyk (1984). Thorough derivations and discussions are carried out from the perspective of modelling and implementation.

GOVERNING EQUATIONS

The starting point is the URANS equations along with the continuity and energy equations. If one writes them in components in the cylindrical coordinates, these equations become

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho V_\theta}{\partial \theta} = 0 \quad (1)$$

$$\frac{\partial \rho V_x}{\partial t} + \frac{\partial \rho V_x V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho V_x V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho V_x V_\theta}{\partial \theta} = -\frac{\partial p}{\partial x} + F_{tx} \quad (2)$$

$$\begin{aligned} \frac{\partial \rho V_r}{\partial t} + \frac{\partial \rho V_r V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho V_r V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho V_r V_\theta}{\partial \theta} - \frac{\rho V_\theta V_\theta}{r} \\ = -\frac{\partial p}{\partial r} + F_{tr} \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial \rho V_\theta}{\partial t} + \frac{\partial \rho V_\theta V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho V_\theta V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho V_\theta V_\theta}{\partial \theta} + \frac{\rho V_r V_\theta}{r} \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + F_{t\theta} \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial \rho h_0}{\partial t} + \frac{\partial \rho h_0 V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho h_0 V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho h_0 V_\theta}{\partial \theta} \\ = \frac{\partial p}{\partial t} + \vec{F}_\tau \cdot \vec{V} + D + Q \end{aligned} \quad (5)$$

$$p = \rho RT \quad (6)$$

Note that the flow in the absolute frame is of interest, the equations for total enthalpy thus replaces the famous rothalpy conservation, which is the energy balance law for the relative flow field. In the energy equation, the term \vec{F}_τ and D are the viscous force and the associated viscous dissipation,

respectively. They can be related to velocity gradients for the common Newtonian fluid and thus bring closure for the equations set. Q is the heat transfer term which, in general, consists of the conduction and convection heat transfers.

TIME AVERAGE FOR ISOLATED ROTORS

At first, one should understand the averaging operators for rotor blade rows and stator blade rows. The first averaging operator to be introduced is the time average operator defined for rotors – for an arbitrary scalar quantity ϕ , the time average operator is defined as

$$\bar{\phi} = \frac{1}{\mathfrak{T}} \int_{\tau}^{\tau+\mathfrak{T}} H \phi dt \quad (7)$$

where the gate function H , accounting for the existence of the blade solid volume in the fluid domain, is defined as

$$H = U(t - \tau) + U\left(t - \tau - \frac{1}{\omega}(\theta - \theta_2)\right) - U\left(t - \tau - \frac{1}{\omega}(\theta - \theta_1)\right) \quad (\text{rotor}) \quad (8)$$

$$H = 1 \quad (\text{outside rotor}) \quad (9)$$

In the gate function, θ_1 and θ_2 are the shape of the pressure surface and suction surface of the blade. These geometrical quantities are defined as Figure 1 illustrates.

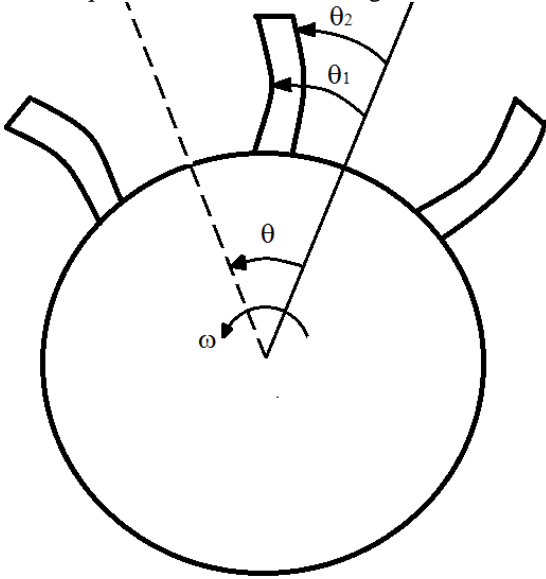


Figure 1: Geometrical Parameters for Rotor

The time average operator is essentially a filtering process which screens out the time-dependent variations and oscillations that have associating time scales smaller than the averaging time interval \mathfrak{T} . For common turbomachinery blade rows, the usual periodic time scales are the blade passing period and the revolutionary period. That is, of course if one only considers the aerodynamic loading in terms of mean flows (i.e., neglecting structural differences from blade to blade and minor disturbances associated with the turbulence). If one sets the average time interval to either one of these two periods, the resultant averaged equations will end up in the steady state. To develop the model for the general 3-D inflow, certain level of circumferential variations need to be

maintained and thus in this paper, the time interval equals to the blade passing period, i.e.,

$$\mathfrak{T} = \frac{2\pi}{\omega N} \quad (10)$$

For further convenience in the derivations, referring to the Favre average (Favre, 1965), one can define an associated time average operator weighted to the density. The density weighted time average is defined as

$$\tilde{\phi} = \frac{\int_{\tau}^{\tau+\mathfrak{T}} \rho H \phi dt}{\int_{\tau}^{\tau+\mathfrak{T}} \rho H dt} \quad (11)$$

Following the definition of the time average operators, the blockage factor needs to be defined for further use. This factor is introduced to account for the effect of flow blocking to due to the blade thickness. It is defined as

$$B = \begin{cases} 1 - \frac{(\theta_1 - \theta_2)N}{2\pi} & (\text{inside rotor}) \\ 1 & (\text{outside rotor}) \end{cases} \quad (12)$$

TIME AVERAGED CONTINUITY AND MOMENTUM EQUATIONS

If one applies these averaging operators on equation (1) through (4) and equation (6), the averaged equations can be obtained as (examples of derivation can be found in the Appendix)

$$\frac{\partial B \bar{\rho} \tilde{V}_x}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \tilde{V}_r}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \tilde{V}_\theta}{\partial \theta} = 0 \quad (13)$$

$$\begin{aligned} \frac{\partial B \bar{\rho} \tilde{V}_x \tilde{V}_x}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \tilde{V}_x \tilde{V}_r}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \tilde{V}_x \tilde{V}_\theta}{\partial \theta} \\ = -\frac{\partial B \bar{p}}{\partial x} + F_{bx} + B \bar{F}_{\tau x} + P_x \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial B \bar{\rho} \tilde{V}_r \tilde{V}_x}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \tilde{V}_r \tilde{V}_r}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \tilde{V}_r \tilde{V}_\theta}{\partial \theta} - \frac{B \bar{\rho} \tilde{V}_\theta \tilde{V}_\theta}{r} \\ = -\frac{\partial B \bar{p}}{\partial r} + F_{br} + B \bar{F}_{\tau r} + P_r \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial B \bar{\rho} \tilde{V}_\theta \tilde{V}_x}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \tilde{V}_\theta \tilde{V}_r}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \tilde{V}_\theta \tilde{V}_\theta}{\partial \theta} + \frac{B \bar{\rho} \tilde{V}_r \tilde{V}_\theta}{r} \\ = -\frac{1}{r} \frac{\partial B \bar{p}}{\partial \theta} + F_{b\theta} + B \bar{F}_{\tau \theta} + P_\theta \end{aligned} \quad (16)$$

$$\bar{p} = \bar{\rho} R \tilde{T} \quad (17)$$

From equation (13) through (16), we can see that the averaged equations contain different terms on the right-hand-side of each equation. These terms will be discussed separately regarding their physical interpretations.

Blade (Inviscid) Body Force

These force terms accounting for the pressure force on a blade: F_{bx} , F_{br} and $F_{b\theta}$. The expressions for these body force terms are

$$F_{bx} = \frac{1}{\omega \mathfrak{I}} \left[p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial x} - p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial x} \right] \quad (18)$$

$$F_{br} = \frac{1}{\omega \mathfrak{I}} \left[p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial r} - p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial r} \right] \quad (19)$$

$$F_{b\theta} = \frac{1}{\omega r \mathfrak{I}} \left[p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_1) \right) - p \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_2) \right) \right] \quad (20)$$

From these expressions, we can see that the inviscid body forces are related to the pressure distributions on the pressure and suction surfaces of a blade and the geometries of both surfaces. The interpretation is very intuitive as the pressure difference between both surfaces serves as the circumferential pressure loading acting on a blade. The axial and radial pressure loading, on the other hand, depends on the shape of the blade surfaces. Mathematically, these terms originate from the gate function defined in equation (8) and equation (9). It is noteworthy that these BF formulations are also valid in axisymmetric throughflow models, as in the work of Jennions and Stow (1985a,b) and Simon and Léonard (2005).

Viscous Terms

The terms \bar{F}_{tx} , \bar{F}_{tr} , and $\bar{F}_{t\theta}$ represent effects of viscosity upon the momentum balance. In the presence of physical viscosity, they equal to the time averaged value of the divergence of viscous stress, which is proportional to the gradient of velocity in Newtonian flow. From modelling perspective, on the other hand, these terms can also be regarded body force terms. Bosman & Marsh (1974) proposed to relate these viscous body force terms to the gradient of entropy inside the body force region. The detailed expressions of these terms are as follows

$$\vec{F}_t = \begin{bmatrix} F_{t,x} \\ F_{t,r} \\ F_{t,\theta} \end{bmatrix} = \begin{bmatrix} -\bar{\rho} \mathcal{L}(x, r, \theta) \frac{\vec{\bar{W}}}{|\vec{\bar{W}}|} \cdot \vec{e}_x \\ -\bar{\rho} \mathcal{L}(x, r, \theta) \frac{\vec{\bar{W}}}{|\vec{\bar{W}}|} \cdot \vec{e}_r \\ -\bar{\rho} \mathcal{L}(x, r, \theta) \frac{\vec{\bar{W}}}{|\vec{\bar{W}}|} \cdot \vec{e}_\theta \end{bmatrix} \quad (21)$$

$$\mathcal{L}(x, r, \theta) = \frac{\tilde{T}}{|\vec{\bar{W}}|} \vec{\bar{W}} \cdot \nabla \tilde{s} \quad (22)$$

Perturbation Terms

Besides the body force terms, certain extra terms, P_x , P_r and P_θ are also present in equation (13) through (16). These terms are derived to be

$$P_i = \frac{\partial B \bar{\rho} \widetilde{V}_i'' \widetilde{V}_i''}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \widetilde{V}_i'' \widetilde{V}_r''}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \widetilde{V}_i'' \widetilde{V}_\theta''}{\partial \theta} - \delta_{ir} \frac{B \bar{\rho} \widetilde{V}_\theta'' \widetilde{V}_\theta''}{r} + \delta_{i\theta} \frac{B \bar{\rho} \widetilde{V}_r'' \widetilde{V}_\theta''}{r} \quad (23)$$

$(i = x, r, \theta)$

These terms came from the averaging the non-linear terms in the original URANS equations. They represent the perturbations of unsteadiness in the flow and it requires extra information to reach a closure description. One can derive and see that these perturbations are equivalent to those in Jennions & Stow (1985a) if the coordinate system is translated from absolute to relative under the axisymmetric precondition. Physically, they are the quantified descriptions of the fluctuations of flow convections. In order to quantitatively describe these terms, in addition to throughflow equations, one also needs flow fields on blade-to-blade surfaces and constant-axial surfaces.

Nonetheless, some precedent researches have shown that these perturbation terms are usually small in magnitude compared to the mean flow parameters (Jennions & Stow, 1985b; Hathaway et al., 1987). For simplicity, these terms can be neglected in general throughflow analyses.

TIME AVERAGED ENERGY EQUATION

Energy conservation can be written in terms of the balance of total enthalpy, i.e., equation (5). However, the definition of the total enthalpy (equation (24)) makes it very difficult to perform the time average operation over the energy equation due to the nonlinearity of the $\frac{1}{2} |\vec{V}|^2$ term.

$$h_0 = h + \frac{1}{2} |\vec{V}|^2 \quad (24)$$

In order to carry out the governing energy equation for the body force model, the total enthalpy is decomposed as follows (using Einstein index notation)

$$\left. \begin{aligned} h_0 &= \tilde{h}_0 + h_{02} + h_0'' \\ \tilde{h}_0 &= \tilde{h} + \frac{1}{2} \tilde{V}_i \tilde{V}_i \\ h_{02} &= \tilde{V}_i V_i'' \\ h_0'' &= h'' + \frac{1}{2} V_i'' V_i'' \end{aligned} \right\} \quad (25)$$

Essentially, the original total enthalpy was expressed as the summation of the “time averaged” total enthalpy, the cross kinetic energy and the perturbed total enthalpy. Since the body force model eventually focuses on the averaged flow field, the governing equation for the “time averaged” total enthalpy, namely \tilde{h}_0 , is of the greatest interest. Applying the time average operator upon the energy equation (Derivation in the Appendix), one can obtain the energy equations for \tilde{h}_0 as follows

$$\begin{aligned}
& \frac{\partial B \bar{\rho} \tilde{h}_0 \tilde{V}_x}{\partial x} + \frac{1}{r} \frac{\partial r B \bar{\rho} \tilde{h}_0 \tilde{V}_r}{\partial r} + \frac{1}{r} \frac{\partial B \bar{\rho} \tilde{h}_0 \tilde{V}_\theta}{\partial \theta} \\
& = + \tilde{V}_x (F_{bx} + B \bar{F}_{\tau x} + P_x) \\
& \quad + \tilde{V}_r (F_{br} + B \bar{F}_{\tau r} + P_r) \\
& \quad + \tilde{V}_\theta (F_{b\theta} + B \bar{F}_{\tau \theta} + P_\theta) \\
& + \bar{\rho} B \bar{T} \left(\tilde{V}_x \frac{\partial \tilde{s}}{\partial x} + \tilde{V}_r \frac{\partial \tilde{s}}{\partial r} + \frac{\tilde{V}_\theta}{r} \frac{\partial \tilde{s}}{\partial \theta} \right) \\
& - \bar{p} \left(\tilde{V}_x \frac{\partial B}{\partial x} + \tilde{V}_r \frac{\partial B}{\partial r} + \frac{\tilde{V}_\theta}{r} \frac{\partial B}{\partial \theta} \right) \quad (26)
\end{aligned}$$

where

$$\nabla \tilde{s} = c_v \frac{\nabla \bar{T}}{\bar{T}} - R \frac{\nabla \bar{p}}{\bar{p}} \quad (27)$$

According to equation (26), in the time averaged flow field, the total enthalpy change is related to 1) the work done by the momentum sources, namely, F_{bi} , $\bar{F}_{\tau i}$ and P_i , 2) entropy change along the streamline and 3) the blockage effect, which is introduced by the effective thickness of a blade.

However, it needs to be stressed that the averaged entropy is also defined in terms of time averaged quantities, just as how the average total enthalpy was defined. One can easily show that the average entropy in equation (27) is different from directly averaging the entropy in URANS flow fields. This implies that in order to properly validate and apply this model, the pre-processing of each URANS/instantaneous quantity must strictly obey the definition of it.

FINAL TIME AVERAGED EQUATIONS

Let us summarize the equations that are derived in previous sections. Given that the perturbation terms are negligible, the final intrinsic form of the time averaged equations, as well as the governing equations for the 3D body force model, is as follows

$$\nabla \cdot (B \bar{\rho} \vec{V}) = 0 \quad (28)$$

$$\bar{\rho} \vec{V} \cdot \nabla \vec{V} = -\frac{1}{B} \nabla (B \bar{p}) + \frac{\vec{F}_b}{B} + \vec{F}_\tau \quad (29)$$

$$\bar{\rho} \vec{V} \cdot \nabla \tilde{h}_0 = \frac{\vec{V} \cdot \vec{F}_b}{B} + \vec{V} \cdot \vec{F}_\tau + \bar{\rho} \bar{T} \vec{V} \cdot \nabla \tilde{s} - \bar{p} \vec{V} \cdot \frac{\nabla B}{B} \quad (30)$$

The left-hand-sides of these equations bear the same form as that of the steady RANS equations, i.e., the steady form of equation (1) through (6). This provides users of this model convenience to implement it into existing steady RANS solvers. The key procedure is to properly formulate the body force terms and the blockage term B .

MOVING (SPACE) AVERAGED EQUATIONS FOR ISOLATED STATORS

Analogously, one can define a pair of space average operators for an isolated stator row. Following the concept of moving average (Hyndman, 2009), one can define the circumferential space average operator as

$$\hat{\phi} = \frac{N_s}{2\pi B_s} \int_{\vartheta}^{\vartheta + \frac{2\pi}{N_s}} G \phi d\theta \quad (31)$$

The corresponding density-weighted moving average operator is defined as

$$\check{\phi} = \frac{\int_{\vartheta}^{\vartheta + \frac{2\pi}{N_s}} \rho G \phi d\theta}{\int_{\vartheta}^{\vartheta + \frac{2\pi}{N_s}} \rho G d\theta} \quad (32)$$

where G is the gate function defined in equation (33) with geometrical quantities defined in Figure 2.

$$\left. \begin{aligned} G &= U(\theta - \vartheta) + U(\theta - \vartheta - \theta_4) - U(\theta - \vartheta - \theta_3)(stator) \\ G &= 1 (outside stator) \end{aligned} \right\} \quad (33)$$

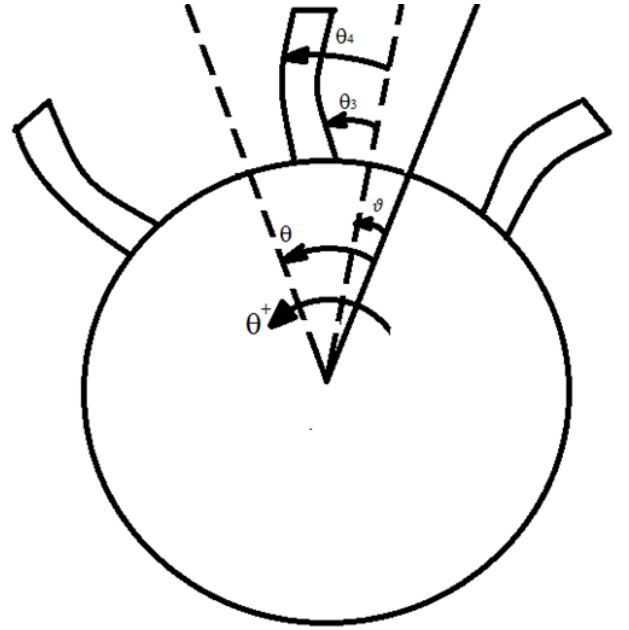


Figure 2: Geometrical Parameters for Stator

The averaging process can be interpreted in two ways: 1) a smoothing technique that screens out circumferential variations with associated circumferential length scale smaller than one blade passage; 2) to average the effect of one single blade upon the nearby flow field within the circumferential distance of one blade passage. Either way, it preserves the circumferential variations of flow quantities from one passage to another. Mathematically, it is equivalent to the averaging operator introduced in Jennions & Stow (1985a).

Considering that for an isolated stator with steady inflow, the flow field can be approximated as steady in the absolute frame, the governing equations are essentially the steady form of equations (1) through (6). One can apply the moving average operators upon the steady form of equations (1) through (6). Defining the blockage factor (equation (34)) for a stator stage, the resultant moving averaged equations are shown in equation (35) through (37)

$$B = \begin{cases} 1 - \frac{(\theta_4 - \theta_3)N_s}{2\pi} (inside\ stator) \\ 1 (outside\ stator) \end{cases} \quad (34)$$

$$\nabla \cdot (B\hat{\rho}\vec{V}) = 0 \quad (35)$$

$$\hat{\rho}\vec{V} \cdot \nabla\vec{V} = -\frac{1}{B}\nabla(B\hat{p}) + \frac{\vec{F}_{bs}}{B} + \vec{F}_\tau + \vec{F}_s \quad (36)$$

$$\hat{\rho}\vec{V} \cdot \nabla\check{h}_0 = \frac{\vec{V} \cdot \vec{F}_{bs}}{B} + \vec{V} \cdot \vec{F}_\tau + \rho\check{T}\vec{V} \cdot \nabla\check{s} - \hat{p}\vec{V} \cdot \frac{\nabla B}{B} \quad (37)$$

The body force terms and perturbation terms, which are neglected eventually, are derived to be following:

$$F_{bsx} = \frac{N_s}{2\pi} \left[p(x, r, \vartheta + \theta_3) \frac{\partial\theta_3}{\partial x} - p(x, r, \vartheta + \theta_4) \frac{\partial\theta_4}{\partial x} \right] \quad (38)$$

$$F_{bsr} = \frac{N_s}{2\pi} \left[p(x, r, \vartheta + \theta_3) \frac{\partial\theta_3}{\partial r} - p(x, r, \vartheta + \theta_4) \frac{\partial\theta_4}{\partial r} \right] \quad (39)$$

$$F_{bs\theta} = \frac{N_s}{2\pi r} [p(x, r, \vartheta + \theta_4) - p(x, r, \vartheta + \theta_3)] \quad (40)$$

$$P_{si} = \frac{\partial B_s \hat{\rho} V_i'''' V_x''''}{\partial x} + \frac{1}{r} \frac{\partial r B_s \hat{\rho} V_i'''' V_r''''}{\partial r} + \frac{1}{r} \frac{\partial B_s \hat{\rho} V_i'''' V_\theta''''}{\partial \theta} - \delta_{ir} \frac{B_s \hat{\rho} V_\theta'''' V_\theta''''}{r} + \delta_{i\theta} \frac{B_s \hat{\rho} V_r'''' V_\theta''''}{r} \quad (41)$$

($i = x, r, \theta$)

FINAL EQUATIONS

To combines the averaged equations for both isolated rotors and isolated stators, one simply needs to apply the time average operator and the moving average operator consequently onto the URANS equation. Assuming all the perturbations are negligible, with extended blockage definition in equation (42), the resultant equations are as follows

$$B = \begin{cases} 1 - \frac{(\theta_1 - \theta_2)N}{2\pi} & (\text{inside rotor}) \\ 1 - \frac{(\theta_4 - \theta_3)N_s}{2\pi} & (\text{inside stator}) \\ 1 & (\text{outside stator}) \end{cases} \quad (42)$$

$$\frac{\partial B \hat{\rho}}{\partial t} + \nabla \cdot (B \hat{\rho} \vec{V}) = 0 \quad (43)$$

$$\hat{\rho} \frac{\partial B \vec{V}}{\partial t} + \hat{\rho} \vec{V} \cdot \nabla \vec{V} + \frac{1}{B} \nabla (B \hat{p}) = \frac{\vec{F}_b}{B} + \vec{F}_\tau \quad (44)$$

$$\hat{\rho} \frac{\partial B \check{e}_t}{\partial t} + \hat{\rho} \vec{V} \cdot \nabla \check{h}_0 = S_e \quad (45)$$

$$S_e = \begin{cases} \frac{\vec{V} \cdot \vec{F}_b}{B} + \vec{V} \cdot \vec{F}_\tau + \rho\check{T}\vec{V} \cdot \nabla\check{s} - \hat{p}\vec{V} \cdot \frac{\nabla B}{B} & (\text{rotor}) \\ 0 & (\text{stator}) \end{cases} \quad (46)$$

These equations are for a rotor-stator stage. The unsteady terms on the left-hand-side of these equation represents rotor-stator interactions that associate with time scales larger than the time averaging period. Generally speaking, in order to reach the final form of steady equations, there are two ways to address these terms:

1) They can be modelled as acoustic sources that influence the pressure field. Note that the associated quantities

are time averaged, hence the corresponding acoustic model can only handle tonal noise. The details regarding acoustic modelling are outside the scope of this paper and interested reader can refer to Envia et al. (2004) for more information.

2) Select a time average period that yield steady equations out of the time average process. By doing so, the basic structure of the resultant equations will remains the same. The blockage factor, however, will need to be re-defined accordingly to the averaging time scale, rendering different pressure field and energy sources in equations (44) and (46) respectively.

If we compare the equations for rotors, stators and rotor-stator stage, we can see that they are all in the same form. Hence, we define the final governing equations for the body force model in the intrinsic form as:

$$\nabla \cdot (B\rho\vec{V}) = 0 \quad (47)$$

$$\rho\vec{V} \cdot \nabla\vec{V} + \nabla(Bp) = \frac{\vec{F}_b}{B} + \vec{F}_\tau \quad (48)$$

$$\rho\vec{V} \cdot \nabla h_0 = \frac{\vec{V} \cdot \vec{F}_b}{B} + \vec{V} \cdot \vec{F}_\tau + \rho\check{T}\vec{V} \cdot \nabla s - \hat{p}\vec{V} \cdot \frac{\nabla B}{B} \quad (49)$$

According to the previous derivations, the quantities in equations (47) through (49) will vary depending on the type of the blade row they are addressing. That is,

- 1) The quantities are time averaged if these equations are applied onto an isolated rotor;
- 2) The quantities are moving averaged if these equations are applied onto an isolated stator;
- 3) The quantities are time averaged + moving averaged if these equations are applied onto a rotor-stator stage.

DISCUSSIONS

Alternative Body Force Expressions

The 3D body force is formulated in terms of pressure and blade surface. An alternative way of formulate the body force is to relate it to the change of angular momentum. It is particularly useful in inverse design since angular momentum can be related to total temperature via the conservation of rothalpy. Follow the idea of Damle et al. (1997) and one can easily derive the body force formulation from equations (47) through (49). The circumferential component of the body force is

$$rF_{b\theta} + rBF_{\tau\theta} = \nabla \cdot B\rho\vec{V}(rV_\theta) + \frac{\partial Bp}{\partial \theta} \quad (50)$$

The first term on the right-hand-side of equation (50) is the axisymmetric part of the body force. It indicates that the body force is changing the angular momentum of the flow entering the blade row. It is a very intuitive refection of the principle of common turbomachinery. The second term, $\frac{\partial Bp}{\partial \theta}$, represents the 3D effect within the flow. More specifically, this term represents the coupling of pressure field on the circumferential direction from one blade passage to another. For compact compressors, Longley and Greitzer (1989) have deduced that such circumferential coupling is not of great

importance. Hence precedent researches about modelling on inlet distortion problems usually neglect this term, rendering the very famous parallel compressor model (Mazzaway, 1977). It is also noteworthy that even though the governing equations and the BF expression are 3D, the nature of throughflow is still to evaluate the average influence of one blade passage upon the flow at a given circumferential location. The only difference is that now the average influence becomes 3D under inlet distortions instead of being axisymmetric under clean inflows.

Next, one needs to have the expressions for the other two components of the body force. Applying the concept of mean stream surface -- S_{2m} (Wu, 1952), one can obtain the kinematic relation between these three components (Damle et al., 1997). The S_{2m} is defined in equation (51) and equation (52)

$$S_{2m} = \theta - f(x, r) \quad (51)$$

$$\vec{W} \cdot \nabla S_{2m} = 0 \quad (52)$$

Imposing that the inviscid body force is always perpendicular to the mean stream surface, i.e., equation (53), one can have the expressions for the three components of inviscid body force, shown in equation (54).

$$\left. \begin{aligned} \vec{F}_b \cdot \vec{W} &= 0 \\ \vec{F}_b &= |\vec{F}_b| \cdot \nabla S_{2m} \end{aligned} \right\} \quad (53)$$

$$\vec{F}_b = \begin{bmatrix} F_{b,x} \\ F_{b,r} \\ F_{b,\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial f}{\partial x} F_{b,\theta} \\ -\frac{\partial f}{\partial r} F_{b,\theta} \\ F_{b,\theta} \end{bmatrix} \quad (54)$$

Note that the S_{2m} is usually approximated as the camber line of a blade with corrections to deviations. Hence one can take blade geometry and extract the camber surface. The camber surface can be input into the equations get performance data in analysis tasks and vice versa in inverse tasks.

Corrections to Loss Model

From equation (50), one can see that in order to acquire explicit expressions for the body force, one still needs to formulate the viscous body force term, $F_{\tau\theta}$. We have introduced in the previous section that the viscous body force can be related to entropy change across a blade row, i.e., equation (21) and equation (22). This loss model was formulated in the relative frame of motion. Since we formulated throughflow equations in the absolute equations, the loss model therefore requires certain corrections. From equations (47) through (49), the corresponding Crocco's equation writes as

$$\vec{V} \times (\nabla \times \vec{V}) = \rho \nabla h_0 - \rho T \nabla s + p \frac{1}{B} \nabla(B) - \frac{\vec{F}_b}{B} - \vec{F}_\tau \quad (55)$$

Based on the work of Lyman (1993), if we assume that all the viscous dissipation is drained by the heat conduction on the blade surfaces (Denton, 1986), then by $\vec{V} \cdot$ (55), we can deduce that

$$\rho \vec{V} \cdot \nabla h_0 = \omega r \left(\frac{F_{b\theta}}{B} + F_{\tau\theta} \right) - \frac{p \vec{V} \cdot \nabla B}{B} \quad (56)$$

Comparing equation (56) to equation (49), one can clearly see that the viscous body force must satisfy

$$\vec{W} \cdot \vec{F}_\tau = -\rho T \vec{V} \cdot \nabla s \quad (57)$$

Following how Bosman and Marsh (1974) formulate their model, the expression for 3D loss model becomes

$$\vec{F}_\tau = \begin{bmatrix} F_{\tau,x} \\ F_{\tau,r} \\ F_{\tau,\theta} \end{bmatrix} = \begin{bmatrix} -\rho \mathcal{L}(x, r, \theta) \frac{\vec{W}}{|\vec{W}|} \cdot \vec{e}_x \\ -\rho \mathcal{L}(x, r, \theta) \frac{\vec{W}}{|\vec{W}|} \cdot \vec{e}_r \\ -\rho \mathcal{L}(x, r, \theta) \frac{\vec{W}}{|\vec{W}|} \cdot \vec{e}_\theta \end{bmatrix} \quad (58)$$

$$\mathcal{L}(x, r, \theta) = \frac{T}{|\vec{W}|} \vec{V} \cdot \nabla s \quad (59)$$

One can see that the loss body force in 3D throughflow model is of the same form as that in axisymmetric model (i.e., equation (58) versus equation (21)). It is the magnitude where the 3D loss model differs from the conventional one (i.e., equation (59) versus equation (22)). Such a difference indicates that the extra viscous force must be introduced into the flow in order to account for the circumferential variation of entropy field. If the inflow is clean, the 3D loss model will reduce to the conventional loss model by neglecting the circumferential variation of entropy.

CONCLUSION

The 3D body force model was thoroughly derived from the URANS equations. The resultant governing equations and body force are full 3D and therefore they are theoretically suitable for general 3D problems.

Some highlights regarding the model are summarized here. First of all, the energy equation in 3D throughflow was also derived rigorously. It demonstrated the missing part in energy conservation when utilizing steady throughflow model for general turbomachine analysis. Similar issue will occur in momentum equations if users of this model cannot put the perturbation terms to closure, which requires calculations in different planes, such as that in blade-to-blade surfaces.

Secondly, the derivation provides users of this model requirements regarding data pre-processing. For an isolated rotor or stator, the model requires input data are of the type of time averaged or moving averaged, respectively. For rotor-stator stage, the input data must be time average and moving average in sequence. Yet extra terms accounting for rotor-stator interactions are carried out in unsteady form from the compounded averaging process. These terms need extra modelling. But they are of the author's intense interest as they can provide ports to implement acoustic analysis into conventional throughflows.

Thirdly, the alternative body force expression demonstrated that the 3D throughflow model takes the

circumferential coupling of the blade passages into consideration. Besides the inviscid body force, modifications were made to the conventional loss model to extend its usage to 3D throughflow. The resultant form of 3D body force is of easy implementation. Especially for CFD based throughflow analysis, such formulations are particularly useful.

In all, this paper introduced the theoretical basis of 3D throughflow and demonstrated its potential. The next phase is to validate the derived 3D body force model, especially in general inlet distortion problems.

NOMENCLATURE

B	=	blockage factor
D	=	viscous dissipation
δ	=	Dirac function
F	=	body forces
f	=	S_{2m} surface function
h	=	enthalpy
\mathcal{L}	=	loss body force magnitude
n	=	normal direction
P	=	perturbation
p	=	pressure
ρ	=	density
Q	=	heat transfer
θ	=	circumferential coordinates
R	=	specific gas constant
r	=	radial coordinates
S	=	source terms
S_{2m}	=	mean S_2 surface
s	=	entropy
T	=	temperature
τ	=	reference time
ϑ	=	reference circumferential location
V	=	absolute velocity
W	=	relative velocity
ω	=	rotational speed
x	=	axial coordinates

Superscripts

-	=	time average
\sim	=	ρ -weighted time average
\wedge	=	moving average
\smile	=	ρ -weighted moving average
\frown	=	time average fluctuation
$\smile\smile$	=	ρ -weighted time average fluctuation
$\smile\smile\smile$	=	moving average fluctuation
$\smile\smile\smile\smile$	=	ρ -weighted moving average fluctuation

Subscripts

0	=	stagnation quantities
b	=	blade body force
i	=	free index (i = x, r, θ)
s	=	quantities at the OGV or stator
θ	=	in circumferential direction
r	=	in radial direction
τ	=	viscous (loss) body force

x = in axial direction

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APPENDIX A: TIME AVERAGE (MOVING AVERAGE) PROCESS

Note that even though the following derivations are performed for the time average operator, they can be extended analogously to the moving average operator.

At first, some basic definitions will be introduced. For an arbitrary quantity Φ inside a rotor, it can always be decomposed as

$$\Phi = \bar{\Phi} + \Phi' = \tilde{\Phi} + \Phi'' \quad (\text{A.1-1})$$

where

$$\bar{\Phi}' = \tilde{\Phi}'' = 0 \quad (\text{A.1-2})$$

Since we are aiming at a set of steady equations at the end of the day, we assume that the flow field within a rotating blade row is periodic over the blade passing period, i.e.,

$$\Phi(t) = \Phi(t + \mathfrak{T}) \quad (\text{A.1-3})$$

where \mathfrak{T} is usually one blade passing period for an isolated rotor.

Next, we need to derive how the time average operator interacts with different derivative operators. Starting with the time derivative, according to the definition of time average, one can have

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial t} &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} H \frac{\partial \Phi}{\partial t} dt \\ &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \frac{\partial H\Phi}{\partial t} dt \\ &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \Phi \frac{\partial H}{\partial t} dt \quad (\text{A.1-4}) \end{aligned}$$

The first underscored term equals to zero since we assume that the flow field is periodic over \mathfrak{T} . As for the second underscored term,

$$\begin{aligned} \frac{\partial H}{\partial t} &= \delta(t - \tau) + \left[\delta \left(t - \tau - \frac{1}{\omega}(\theta - \theta_2) \right) \right. \\ &\quad \left. - \delta \left(t - \tau - \frac{1}{\omega}(\theta - \theta_1) \right) \right] \quad (\text{A.1-5}) \end{aligned}$$

Then it can be obtained that

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial t} &= \frac{1}{\mathfrak{T}B} \left[\Phi \left(x, r, \theta, \tau + \frac{1}{\omega}(\theta - \theta_1) \right) \right. \\ &\quad \left. - \Phi \left(x, r, \theta, \tau + \frac{1}{\omega}(\theta - \theta_2) \right) \right] \quad (\text{A.1-6}) \end{aligned}$$

With regards to the spatial derivatives, taking radial component as an example:

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial r} &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} H \frac{\partial \Phi}{\partial r} dt \\ &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \frac{\partial H\Phi}{\partial r} dt \\ &= \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \Phi \frac{\partial H}{\partial r} dt \quad (\text{A.1-7}) \end{aligned}$$

For the first underscored term, the radial derivative is interchangeable with time integral, and thus

$$\begin{aligned} \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \frac{\partial H\Phi}{\partial r} dt &= \frac{1}{B} \frac{\partial}{\partial r} B \frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} H\Phi dt \\ &= \frac{1}{B} \frac{\partial B\bar{\Phi}}{\partial r} \quad (\text{A.1-8}) \end{aligned}$$

As for the second underscored term,

$$\begin{aligned} \frac{\partial H}{\partial r} &= \frac{1}{\omega} \left[\delta \left(t - \tau - \frac{1}{\omega}(\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial r} \right. \\ &\quad \left. - \delta \left(t - \tau - \frac{1}{\omega}(\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial r} \right] \quad (\text{A.1-9}) \end{aligned}$$

Then then second underscored term in equation (A.1-7) becomes

$$\begin{aligned} &\frac{1}{\mathfrak{T}B} \int_{\tau}^{\tau+\mathfrak{T}} \Phi \frac{\partial H}{\partial r} dt \\ &= \frac{1}{\omega \mathfrak{T}B} \left[\Phi \left(\tau + \frac{1}{\omega}(\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial r} \right. \\ &\quad \left. - \Phi \left(\tau + \frac{1}{\omega}(\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial r} \right] \quad (\text{A.1-10}) \end{aligned}$$

Therefore, interchanging the time-average and radial derivative, one can have

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial r} &= \frac{1}{B} \frac{\partial B\bar{\Phi}}{\partial r} + \frac{1}{\omega \mathfrak{T}B} \left[\Phi \left(x, r, \theta, \tau + \frac{1}{\omega}(\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial r} \right. \\ &\quad \left. - \Phi \left(x, r, \theta, \tau + \frac{1}{\omega}(\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial r} \right] \quad (\text{A.1-11}) \end{aligned}$$

Similarly, for axial and circumferential derivatives, one can have

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial x} &= \frac{1}{B} \frac{\partial B \bar{\Phi}}{\partial x} + \frac{1}{\omega \mathfrak{I} B} \left[\Phi \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_1) \right) \frac{\partial \theta_1}{\partial x} \right. \\ &\quad \left. - \Phi \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_2) \right) \frac{\partial \theta_2}{\partial x} \right] \end{aligned} \quad (\text{A.1 - 12})$$

$$\begin{aligned} \frac{\partial \bar{\Phi}}{\partial \theta} &= \frac{1}{B} \frac{\partial B \bar{\Phi}}{\partial \theta} + \frac{1}{\omega \mathfrak{I} B} \left[\Phi \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_2) \right) \right. \\ &\quad \left. - \Phi \left(x, r, \theta, \tau + \frac{1}{\omega} (\theta - \theta_1) \right) \right] \end{aligned} \quad (\text{A.1 - 13})$$

Then one can apply these properties of the time average operator onto a conservative expression, that is

$$\begin{aligned} &\frac{\partial \rho \Phi}{\partial t} + \frac{\partial \rho \Phi V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho \Phi V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho \Phi V_\theta}{\partial \theta} \\ &= \frac{1}{B} \left(\frac{\partial B \rho \Phi V_x}{\partial x} + \frac{1}{r} \frac{\partial B r \rho \Phi V_r}{\partial r} + \frac{1}{r} \frac{\partial B \rho \Phi V_\theta}{\partial \theta} \right) \\ &+ \frac{1}{\omega \mathfrak{I} B} \left[(\rho \Phi) \Big|_{\theta_1} \left(V_x \frac{\partial \theta_1}{\partial x} + V_r \frac{\partial \theta_1}{\partial r} - \frac{V_\theta - \omega r}{r} \right) \Big|_{\theta_1} \right. \\ &\left. - (\rho \Phi) \Big|_{\theta_2} \left(V_x \frac{\partial \theta_2}{\partial x} + V_r \frac{\partial \theta_2}{\partial r} - \frac{V_\theta - \omega r}{r} \right) \Big|_{\theta_2} \right] \end{aligned} \quad (\text{A.1 - 14})$$

The last two terms in the right-hand-side of equation (A.1 - 14) are zero according to flow tangency condition, i.e.,

$$\left(V_x \frac{\partial \theta_{1,2}}{\partial x} + V_r \frac{\partial \theta_{1,2}}{\partial r} - \frac{V_\theta - \omega r}{r} \right) \Big|_{\theta_{1,2}} = 0 \quad (\text{A.1 - 15})$$

Then final form of equation (A.1 - 14) becomes

$$\begin{aligned} &\frac{\partial \rho \Phi}{\partial t} + \frac{\partial \rho \Phi V_x}{\partial x} + \frac{1}{r} \frac{\partial r \rho \Phi V_r}{\partial r} + \frac{1}{r} \frac{\partial \rho \Phi V_\theta}{\partial \theta} \\ &= \frac{1}{B} \left(\frac{\partial B \rho \Phi V_x}{\partial x} + \frac{1}{r} \frac{\partial B r \rho \Phi V_r}{\partial r} + \frac{1}{r} \frac{\partial B \rho \Phi V_\theta}{\partial \theta} \right) \end{aligned} \quad (\text{A.1 - 16})$$

If one applies equation (A.1 - 16) onto equation (1) through (5), it takes very simple algebra to reach equation (13) through (16).

APPENDIX B: TIME (MOVING) AVERAGED ENERGY EQUATION

Decompose the total enthalpy as equation (25) does. Then the URANS energy equations becomes

$$\rho \frac{d \tilde{h}_0}{dt} = -\rho \frac{d h_{02}}{dt} - \rho \frac{d h''_0}{dt} + \frac{\partial p}{\partial t} + \vec{F}_\tau \cdot \vec{V} + D + Q \quad (\text{A.2 - 1})$$

Apply the time average operator onto equation (A.2 - 1) and rearrange it, one can have

$$\begin{aligned} \nabla \cdot (B \bar{\rho} \tilde{h}_0 \vec{V}) &= -B \bar{\rho} \frac{d h_{02}}{dt} - \nabla \cdot (B \bar{\rho} \widetilde{h''_0 \vec{V}}) - \nabla \\ &\cdot (B \bar{\rho} \widetilde{V_i'' V_i''} \vec{V}) + B \frac{\partial \bar{p}}{\partial t} + B \overline{F_\tau \cdot \vec{V}} + B \bar{D} \\ &+ B \bar{Q} \end{aligned} \quad (\text{A.2 - 2})$$

Let us define the kinetic energy term E as follows

$$\left. \begin{aligned} E &= \frac{1}{2} V_i V_i \\ \tilde{E} &= \frac{1}{2} \tilde{V}_i \tilde{V}_i \end{aligned} \right\} \quad (\text{A.2 - 3})$$

Then one can easily acquire following relations. The left part in equation (A.2 - 4) comes from the averaged momentum equations while the right part comes from the time average over the unsteady Reynolds Averaged kinetic energy equation.

$$\begin{aligned} &\vec{V} \cdot (\vec{F}_b + B \vec{F}_\tau + \vec{P}) - \vec{V} \cdot \nabla B \bar{p} \\ &= \nabla \cdot (B \bar{\rho} \vec{E} \vec{V}) \\ &= -B \bar{\rho} \frac{d h_{02}}{dt} - \nabla \cdot (B \bar{\rho} \widetilde{V_i'' V_i''} \vec{V}) \\ &\quad - (\vec{V} \cdot \nabla B \bar{p} - B \overline{F_\tau \cdot \vec{V}}) \end{aligned} \quad (\text{A.2 - 4})$$

Substituting equation (A.2 - 4) into equation (A.2 - 2), one can have

$$\begin{aligned} \nabla \cdot (B \bar{\rho} \tilde{h}_0 \vec{V}) &= \vec{V} \cdot (\vec{F}_b + B \vec{F}_\tau + \vec{P}) - \vec{V} \cdot \nabla B \bar{p} - \nabla \\ &\cdot (B \bar{\rho} \widetilde{h''_0 \vec{V}}) + B \left(\frac{d \bar{p}}{dt} + D + Q \right) \end{aligned} \quad (\text{A.2 - 5})$$

Using basic thermodynamics, equation (A.2 - 5) can be rewritten as

$$\begin{aligned} \nabla \cdot (B \bar{\rho} \tilde{h}_0 \vec{V}) &= \vec{V} \cdot (\vec{F}_b + B \vec{F}_\tau + \vec{P}) - \vec{V} \cdot \nabla B \bar{p} + B \left(\rho \frac{d \bar{h}}{dt} \right) \\ &- \nabla \cdot (B \bar{\rho} \widetilde{h''_0 \vec{V}}) \end{aligned} \quad (\text{A.2 - 6})$$

Considering the time average for enthalpy variation, one can have

$$\nabla \cdot (B \bar{\rho} \tilde{h}_0 \vec{V}) = \vec{V} \cdot (\vec{F}_b + B \vec{F}_\tau + \vec{P}) - \vec{V} \cdot \nabla B \bar{p} + \nabla \cdot (B \bar{\rho} \tilde{h} \vec{V}) \quad (\text{A.2 - 7})$$

The second term on the right-hand-side of equation (A.2 - 7) can be further expanded using equation (A.2 - 8) and equation (A.2 - 9)

$$\vec{V} \cdot \nabla B \bar{p} = B \bar{\rho} \vec{V} \cdot \nabla \tilde{h} + B R \vec{T} \vec{V} \cdot \nabla \bar{\rho} - B C_v \bar{\rho} \vec{V} \cdot \nabla \vec{T} + \bar{p} \vec{V} \cdot \nabla B \quad (\text{A.2 - 8})$$

$$B R \vec{T} \vec{V} \cdot \nabla \bar{\rho} - B C_v \bar{\rho} \vec{V} \cdot \nabla \vec{T} = -\bar{\rho} B \vec{T} \vec{V} \cdot \nabla \vec{s} \quad (\text{A.2 - 9})$$

Defining entropy as in equation (27) and substituting equation (A.2 - 8) and equation (A.2 - 9) into equation (A.2 - 7), it yields the governing equation for the average total enthalpy, as in equation (A.2 - 10).

$$\begin{aligned} \nabla \cdot (B \bar{\rho} \tilde{h}_0 \vec{V}) &= \vec{V} \cdot (\vec{F}_b + B \vec{F}_\tau + \vec{P}) \\ &+ \bar{\rho} B \vec{T} \vec{V} \cdot \nabla \vec{s} - \bar{p} \vec{V} \cdot \nabla B \end{aligned} \quad (\text{A.2 - 10})$$

Note that for energy equation in stator, analogous derivations can be performed using moving average operators.